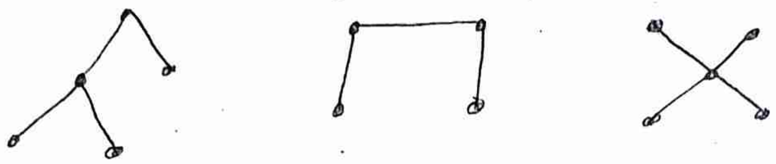
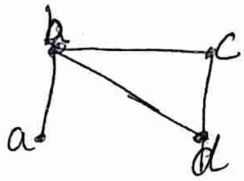
Trees

<u>Definition</u> A Graph G is called a tree if
(i) G is connected
(ii) G has no cycle

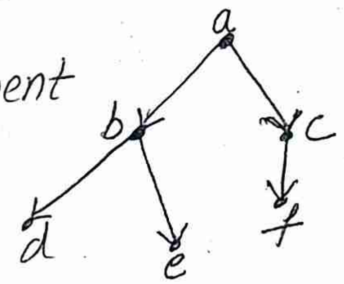
<u>E.g</u>



Graph  is not a tree as it contains a cycle b, c, d, b.

<u>Remark</u>: (i) Tree is said to be directed if every edge of tree is assigned a direction, otherwise tree is undirected.
(ii) Also, a tree has to be simple graph.

<u>Terminology used in tree</u> :-



1) <u>Node or Vertex</u> :- Node is key component of tree which stores information & can have one or more links for connecting to other nodes.

2) <u>Edge or Link</u> :- A directed line from one node to other node is called edge or link or branch of tree.
<u>e.g</u> ab, ac, bd etc.

3) <u>Root</u> :- The vertex having indegree zero is called root of the tree. <u>e.g</u> a is root of above tree.

4) <u>Path</u> :- A path is a sequence of nodes when we traverse from one node to other along the edges which connect them <u>e.g</u> path from a to f is a, c, f.

5) <u>Level</u> :- Level of node is an integer value that measures the distance of a node from the root. Root is at level 0. The child(s) of root are at level 1 & so on

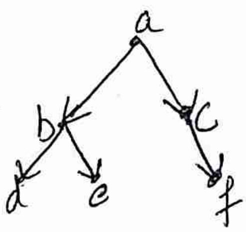
7 **Height**:- Height of node is the length of longest path from node to a leaf. All leaves are at a height 0. Height of root is height of tree. In above tree, height of d, e, f is 0, Height of b, c is 1 & height of a is 2.

7 **Depth**:- The depth of node is the length of ~~longest~~ path from node to root of tree. Root has depth 0.

In above tree depth of 'a' is 0, depth of b, c is 1. Depth of d, e, f is 2.

8 **Rooted tree**:- A rooted tree is a directed tree which contains a unique vertex 'r' s.t. the in-degree of r is zero & every other vertex has in-degree one. The vertex 'r' is called root of rooted Tree.

e.g



is rooted tree with root 'a'.

9 **Parent & Offsprings**:- If (x, y) is any directed edge then x is called Parent of y & y is called offspring of x. Root of tree has no parent whereas every other node has a unique parent. A parent can have several offsprings. Offspring is also called child or son.

In above tree a is parent of b & c. b has two offsprings d & e.

10 **Leaf**: A node having no offsprings (outdegree=0) is called a leaf. In fig. d, e, f are leaves. Leaf is also called external or terminal node.

11 **Siblings**:- Two nodes having same parent are called siblings. In fig. b, c are siblings of a.

12 **Interior node**:- Node having atleast one child is called interior node.

Ancestor: Ancestors of a vertex other than root are the vertices in the path from root to this vertex, excluding the vertex itself & including the root.

Eg:- ancestor of d are b & a

(14) **Descendant:-** Descendants of a vertex 'V' are those vertices that have 'V' as an ancestor.

E.g. Descendants of b are d & e.

(15) **Subtree:-** If 'a' is any vertex in a tree, the subtree with a as its root is the ~~sug~~ subgraph of tree consisting of a & its descendants are all edges incident to these descendants.

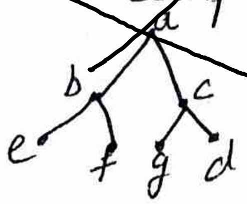
In given fig. subtree of b is T(b) as ~~a~~shown:

(16) **Forest:-** A forest is an undirected graph whose components are all trees.

Binary Tree:- let T is a tree. we say T is n-tree or n-ary tree if every vertex has atmost n offsprings. In particular, if n=2 then tree is called binary tree. So binary tree is that tree in which every node can have 0, 1 or 2 offsprings.

~~**Complete Binary Tree:-** In n-tree, if every vertex of T, other than leaves, has exactly n-offsprings then we say T is complete n-Tree. For n=2, we say tree is complete Binary tree.~~

~~Eg~~



~~is a complete Binary tree~~

~~is complete binary tree~~

Properties of Tree

Property 1: There is one and only one path b/w every pair of vertices in a tree T.

Proof: Since T is connected graph. Therefore there exists atleast one path between every pair of vertices in tree T.

Suppose that between two vertices v_1 & v_2 there exists two distinct paths. The union of these two paths will contain a circuit & then T cannot be a tree. Thus there is one & only one path between every pair of vertices in a tree T.

Property 2: If in a Graph G, there is one & only one path between every pair of vertices, then G is tree.

Proof: Since there is one & only one path between every pair of vertices in G implies G is connected graph. Suppose that G contain a circuit, then there is atleast one pair of vertices v_1, v_2 (say) s.t. there are two distinct path between them. A contradiction to given fact & so G cannot have circuit. Hence G is a connected graph without circuit implies G is a tree.

Property 3: A tree with n vertices has $n-1$ edges.

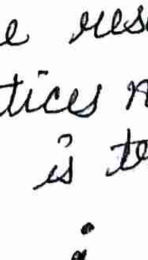
Proof: We shall prove the result by induction on the number of vertices n.

Obviously the result is true for $n=1,2,3$ as

when $n=1$ we have — zero edge

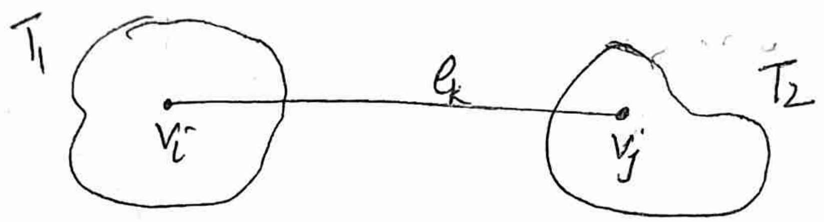
" $n=2$ " — one edge

" $n=3$ " — Two edge



Let us assume result is true for all tree with less than n vertices.

Consider a tree T with n vertices. Let e_k be an edge with end vertices v_i & v_j. Since there is one & only one path between every pair of vertices i.e. there is no other path b/w v_i & v_j except e_k. Therefore deletion of edge from T will disconnect the graph as shown below.

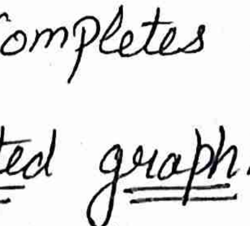


$\therefore$ $T - e_k$ consists of exactly two component T_1 & T_2 (say). Since there is no circuit in T each of these components is a tree. Further, both of these tree T_1 & T_2 have less than n-vertices, $\therefore$ by supposition, each tree will contain edge one less than the number of vertices in it. So $T - e_k$ consists of $(n-2)$ edges implies that T has exactly $(n-2)+1 = n-1$ edges.

This completes the induction.

<u>Minimally connected graph</u>:- A connected graph G is said to be minimally connected if removal of any edge from it disconnect the graph.

<u>Eg</u>



are minimally connected graph.

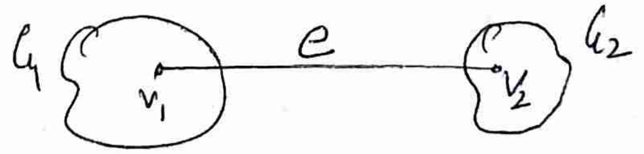
Property 4]:- A graph is a tree iff it is minimally connected.

Proof! Firstly, let the graph T be a tree.
$\therefore$ T must be connected graph.
If possible, let T be not minimally connected.
Then there must exists an edge e_i in T s.t. $T - e_i$ is connected. $\therefore$ e_i is in some circuit.
$\Rightarrow$ T is not a tree, a contradiction.
Hence T must be minimally connected.

Conversely:- Let T be minimally connected graph.
$\therefore$ T cannot have a circuit. otherwise we could remove one of the edge in the circuit & still leave the graph connected.
Hence T is a tree

Property 5] A Graph G with n-vertices & $(n-1)$ edges & no circuit is connected.

Proof! Let there exists a graph with n, $(n-1)$ edges & no circuit which is disconnected. Then G will consists of two or more circuit-less component. without loss of generality, let G consists of two components G_1 & G_2 as shown below:



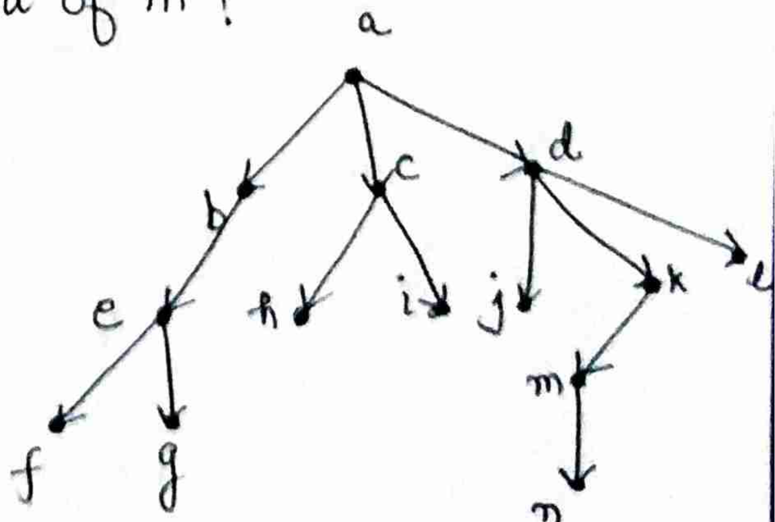
Now add an edge e between vertices v_1 in G_1 & v_2 in G_2.
Since there is no path between v_1 & v_2 in G so by adding

an edge e didnot create a circuit in G. Thus G ∪ e is a circuit less connected graph i.e. In other words G ∪ e is a tree having n-vertices & n edges, which is not possible. Hence a graph with n-vertices & (n−1) edges & no circuit is connected.

Remark:- Five different but equivalent definition of tree are: Graph G with n-vertices is called a tree if

(i) G is connected & has no circuit.
(ii) G is connected & has (n−1) edges.
(iii) G has (n−1) edges & no circuit.
(iv) there is exactly one path between every pair of vertices in G.
(v) G is minimally connected.

Ques:- Consider the tree
a) List all level-3 vertices b) List all leaves
c) What are siblings of of d ? d) Draw the Tree T(b)
e) What is level and height of m ?

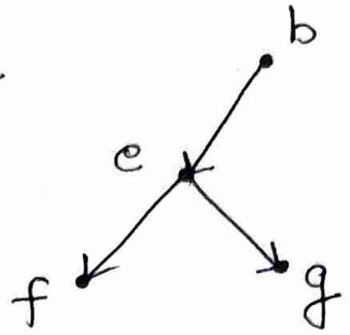


sol:
(a) Root of tree is a
so level of a is 0
" " b,c,d is 1
" " e,h,i,j,k,l is 2
" " f,g,m is 3.

(b) f, g, h, i, j, n and l are leaves as they have no offsprings.

(c) Siblings of d are b and c
$\because$ b, c, d have same parent a.

(d) Tree T(b) is as shown



(e) Level of m = 3
Height of m = 1
Height of tree = 4

Ques:→ In any non-trivial tree, there are atleast two pendent vertices.

Sol:→ Let T be a non-trivial tree with n vertices then T has n−1 edges.
$\therefore$ By Fundamental theorem on graph Theory.

$$\sum_{i=1}^{n} deg(v_i) = 2(n-1) = 2n-2. \quad \text{——— (1)}$$

If possible, let T contain only one vertex of degree 1. Then say v_1 of degree 1.
$deg(v_1) = 1$ and $deg(v_i) \geq 2$ for $i = 2, 3, 4, \ldots, n$.

$$\therefore \sum_{i=1}^{n} deg(v_i) = deg(v_1) + \sum_{i=2}^{n} deg(v_i)$$
$$= 1 + \sum_{i=2}^{n} deg(v_i)$$
$$\geq 1 + 2(n-1) = 2n-1 \quad \text{——— (2)}$$

from (1) & (2) we get

$$2n-2 \geq 2n-1 \quad \rightarrow\leftarrow$$

$\therefore$ T must have more than one vertex of degree 1
i.e. T has atleast two vertices of degree 1.